Topic #2 – Boolean Algebra

- Rules to determine digital logic, a.k.a. `switching algebra'
  - deals with Boolean values – 0, 1
  - Signal values denoted by variables – {X, Y, JAY, …}
- Positive-logic convention
  - analog voltages (LOW, HIGH) → (0, 1)
  - negative logic – seldom used
- Operators: { · , + , ` , ⊕ }
- Axioms and Theorems …
  - helps to reduce complex logic into simpler ones – improve the area and speed of digital circuits.

Boolean expression - definition

- Literal: a variable or its complement
  - X, X′, FRED′, CS_L
- Expression: literals combined by AND, OR, parentheses, complementation
  - X+Y
  - P · Q · R
  - A + B · C
  - ((JAY · Z′) + CS_L · A · B′ · C + Q5) · RESET′
- Equation: variable = expression
  - P = ((JAY · Z′) + CS_L · A · B′ · C + Q5) · RESET′

Axioms

- Axioms
  - minimal set of basic definitions (A1-A5, A1′-A5′) that are assumed to be true and completely define switching algebra.
  - Can be used to prove other switching algebra theorems (T1-T15).

(A1)	X=0, if X≠1	(A1′)	X=1, if X≠0
(A2)	If X=0, then X′=1	(A2′)	If X=1, then X′=0
(A3)	0 · 0 = 0	(A3′)	1 + 1 = 1
(A4)	1 · 1 = 1	(A4′)	0 + 0 = 0
(A5)	0 · 1 = 1 · 0 = 0	(A5′)	1 + 0 = 0 + 1 = 1

Each axiom has a dual

Single-variable theorems (T1-T5)

(T1)	$X + 0 = X$	(T1′)	$X \cdot 1 = X$	(Identities)
(T2)	$X + 1 = 1$	(T2′)	$X \cdot 0 = 0$	(Null elements)
(T3)	$X + X = X$	(T3′)	$X \cdot X = X$	(Idempotency)
(T4)	$(X')' = X$			(Involution)
(T5)	$X + X' = 1$	(T5′)	$X \cdot X' = 0$	(Complements)

- Proved by perfect induction …
  - Since a switching variable can only take the values 0 and 1 we can prove a theorem involving a single variable X by simply plugging in X = 0 and X =1
- Example: (T1) X + 0 = X
  - X=0 : 0 + 0 = 0 ⇒ true according to axiom A4′
  - X=1 : 1 + 0 = 1 ⇒ true according to axiom A5′

Two and three variable theorems (T6-T11)

(T6)	$X + Y = Y + X$	(T6′)	$X \cdot Y = Y \cdot X$	(Commutativity)
(T7)	$(X + Y) + Z = X + (Y + Z)$	(T7′)	$(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$	(Associativity)
(T8)	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$	(T8′)	$(X + Y) \cdot (X + Z) = X + Y \cdot Z$	(Distributivity)
(T9)	$X + X \cdot Y = X$	(T9′)	$X \cdot (X + Y) = X$	(Covering)
(T10)	$X \cdot Y + X \cdot Y' = X$	(T10′)	$(X + Y) \cdot (X + Y') = X$	(Combining)
(T11)	$X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$			(Consensus)
(T11′)	$(X + Y) \cdot (X' + Z) \cdot (Y + Z) = (X + Y) \cdot (X' + Z)$			

- Duality:
  - Swap 0 & 1, AND & OR ⇒ theorems still true?
  - Yes!! … Why? … each axiom has a dual …
- Be careful about `operator precedence' – use parentheses

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Theorems T6, T7

(Commutativity)

(T6) $X + Y = Y + X$

(T6′) $X \cdot Y = Y \cdot X$

(Associativity)

(T7) $(X + Y) + Z = X + (Y + Z)$

(T7′) $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$

- similar to commutative and associative laws for addition and multiplication of integers and real numbers.

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Theorem T8

(Distributivity)

(T8) $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$

(T8′) $(X + Y) \cdot (X + Z) = X + Y \cdot Z$

- sum-of-products vs. product-of-sums

$V \cdot W \cdot Y + V \cdot W \cdot Z + V \cdot X \cdot Y + V \cdot X \cdot Z = V \cdot (W + X) \cdot (Y + Z)$

(sum-of-products form) ***(product-of-sums form)***

$(V \cdot W \cdot X) + (Y \cdot Z) = (V + Y) \cdot (V + Z) \cdot (W + Y) \cdot (W + Z) \cdot (X + Y) \cdot (X + Z)$

- Depending on the problem, choose the one `simpler'
  - Which one is more logical for you?

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Theorems T9, T10

(Covering)

(T9) $X + X \cdot Y = X$

(T9′) $X \cdot (X + Y) = X$

(Combining)

(T10) $X \cdot Y + X \cdot Y' = X$

(T10′) $(X + Y) \cdot (X + Y') = X$

- Useful in simplifying logic functions

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Theorem T11

(Consensus)

(T11) $X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$

(T11') $(X + Y) \cdot (X' + Z) \cdot (Y + Z) = (X + Y) \cdot (X' + Z)$

- In T11 the term $Y \cdot Z$ is called the consensus of the term $X \cdot Y$ and $X' \cdot Z$:
  - If $Y \cdot Z = 0$, then T11 must be true
  - If $Y \cdot Z = 1$, then either $X \cdot Y$ or $X' \cdot Z$ must also be 1
  - Thus the term $Y \cdot Z$ is redundant and may be dropped
- How about (T11')?

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N-variable theorems (T12 – T15)

(T12)	$X + X + \cdots + X = X$	(Generalized idempotency)
(T12')	$X \cdot X \cdot \cdots \cdot X = X$	
(T13)	$(X_1 \cdot X_2 \cdot \cdots \cdot X_n)' = X_1' + X_2' + \cdots + X_n'$	(DeMorgan's theorems)
(T13')	$(X_1 + X_2 + \cdots + X_n)' = X_1' \cdot X_2' \cdot \cdots \cdot X_n'$	
(T14)	$[F(X_1, X_2, \ldots, X_n, +, \cdot)]' = F(X_1', X_2', \ldots, X_n', \cdot, +)$	(Generalized DeMorgan's theorem)
(T15)	$F(X_1, X_2, \ldots, X_n) = X_1 \cdot F(1, X_2, \ldots, X_n) + X_1' \cdot F(0, X_2, \ldots, X_n)$	(Shannon's expansion theorems)
(T15')	$F(X_1, X_2, \ldots, X_n) = [X_1 + F(0, X_2, \ldots, X_n)] \cdot [X_1' + F(1, X_2, \ldots, X_n)]$	

- Prove using finite induction
- Most important: DeMorgan's theorems (T13 & T13')

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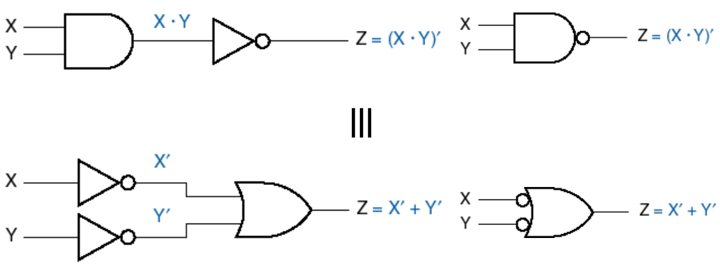
DeMorgan's Theorem Example: NAND

- $(X \cdot Y)' = (X' + Y')$
  - $(X \cdot Y)'$ is typically referred as NAND gate in logic gate expression



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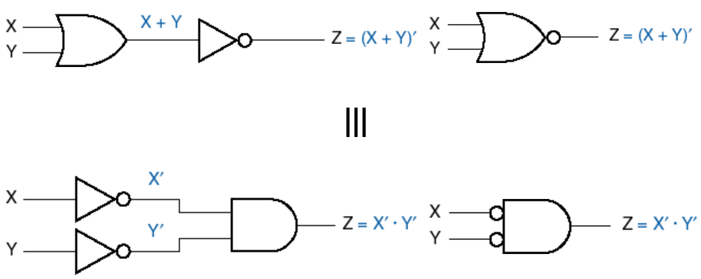
DeMorgan's Theorem Example: NOR

- $(X + Y)' = (X' \cdot Y')$
  - $(X + Y)'$ is typically referred as NOR gate in logic gate expression



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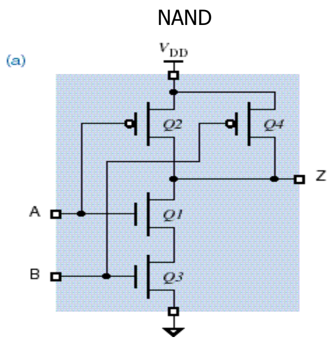
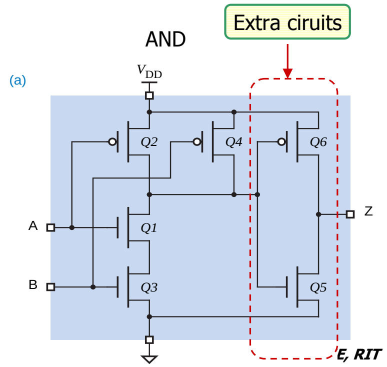
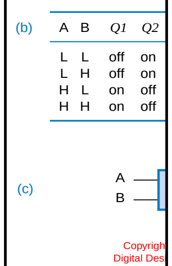
NAND & NOR gates

- Use less number of circuits than AND & OR gates
- Fan-in & Fan-out







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Generalized DeMorgan's theorem

(T14) $[F(X_1, X_2, \ldots, X_n, +, \cdot)]' = F(X_1', X_2', \ldots, X_n', \cdot, +)$

- Given any n-variable logic expression, its complement can be found by swapping + and · and complementing all variables
- Example:
  - F(W,X,Y,Z) = (W′ · X) + (X · Y) + (W · (X′ + Z′))
    = ((W)′ · X) + (X · Y) + (W · ((X)′ + (Z)′))
  - [F(W,X,Y,Z)]′ = ((W′)′ + X′) · (X′ + Y′) · (W′ + ((X′)′ · (Z′)′))
  - Using (T4) (X′)′ = X, above can be simplified to:
  - [F(W,X,Y,Z)]′ = (W + X′) · (X′ + Y′) · (W′ + (X · Z))

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Duality revisited

- Any theorem in switching algebra remains true if 0 & 1 are swapped and · & + are swapped.
- True because all the duals of all the axioms are true, so duals of all switching algebra theorems can be proven using the duals of axioms.
- One can re-write DeMorgan's theorem as

$$[F(X_1, X_2, \ldots, X_n)]' = F^D(X_1', X_2', \ldots, X_n')$$

- Note …
  - A · B + C ≠ A + B · C
    ≠ (A + B) · C
  - Duality does not mean equivalence !!

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Manipulation of Boolean expression

- How can we re-write (A · B + C) then?
- Use DeMorgan's theorem …
  - A · B + C = ((A · B + C)′)′
  - = ((A · B)′ · C′)′
  - = ((A′ + B′) · C′)′
  - ⇨ (A · B + C)′ = (A′ + B′) · C′

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Switching Algebra Axioms & Theorems

(A1) $X = 0$ if $X \neq 1$	(A1′) $X = 1$ if $X \neq 0$	
(A2) If $X = 0$, then $X' = 1$	(A2′) if $X = 1$, then, $X' = 0$	
(A3) $0 . 0 = 0$	(A3′) $1 + 1 = 1$	
(A4) $1 . 1 = 1$	(A4′) $0 + 0 = 0$	
(A5) $0 . 1 = 1 . 0 = 0$	(A5′) $1 + 0 = 0 + 1 = 1$	

(T1) $X + 0 = X$	(T1′) $X . 1 = X$	(Identities)
(T2) $X + 1 = 1$	(T2′) $X . 0 = 0$	(Null elements)
(T3) $X + X = X$	(T3′) $X . X = X$	(Idempotency)
(T4) $(X')' = X$		(Involution)
(T5) $X + X' = 1$	(T5′) $X . X' = 0$	(Complements)
(T6) $X + Y = Y + X$	(T6′) $X . Y = Y . X$	(Commutativity)
(T7) $(X + Y) + Z = X + (Y + Z)$	(T7′) $(X . Y) . Z = X . (Y . Z)$	(Associativity)
(T8) $X . Y + X . Z = X . (Y + Z)$	(T8′) $(X + Y) . (X + Z) = X + Y . Z$	(Distributivity)
(T9) $X + X . Y = X$	(T9′) $X . (X + Y) = X$	(Covering)
(T10) $X . Y + X . Y' = X$	(T10′) $(X + Y) . (X + Y') = X$	(Combining)
(T11) $X . Y + X'. Z + Y . Z = X . Y + X' . Z$		
(T11′) $(X + Y) . (X' + Z) . (Y + Z) = (X + Y) . (X' + Z)$		(Consensus)
(T12) $X + X + \ldots + X = X$	(T12′) $X . X . \ldots . X = X$	(Generalized idempotency)
(T13) $(X_1 . X_2 . \ldots . X_n)' = X_1' + X_2' + \ldots + X_n'$		
(T13′) $(X_1 + X_2 + \ldots + X_n)' = X_1' . X_2' . \ldots . X_n'$		(DeMorgan's theorems)
(T14) $[F(X_1, X_2, \ldots, X_n, +, .)]' = F(X_1', X_2', \ldots, X_n', ., +)$		(Generalized DeMorgran's theorem)

Boolean expression – more definition

- A product term:
  - Z', $(W \cdot X \cdot Y)$, $(X \cdot Y' \cdot Z)$, $(W' \cdot Y' \cdot Z)$
- A sum term:
  - Z', $(W + X + Y)$, $(X + Y' + Z)$, $(W' + Y' + Z)$
- A sum-of-products expression:
  - $Z' + (W \cdot X \cdot Y) + (X \cdot Y' \cdot Z) + (W' \cdot Y' \cdot Z)$
- A product-of-sums expression:
  - $Z' \cdot (W + X + Y) \cdot (X + Y' + Z) \cdot (W' + Y' + Z)$
- A normal term: a product or sum term in which no variable appears more than once

  Examples of non-normal terms: $W \cdot X \cdot X \cdot Y'$ $W+W+X'+Y$ $X \cdot X' \cdot Y$

  Examples of normal terms: $W \cdot X \cdot Y'$ $W+X'+Y$ 0

Minterm and Maxterm

- Minterm:
  - An n-variable minterm is a normal product term with n literals.
  - There are 2^n such products terms.
  - Example of 4-variable minterms:

    $W \cdot X' \cdot Y' \cdot Z'$ $W \cdot X \cdot Y' \cdot Z$ $W' \cdot X' \cdot Y \cdot Z'$
  - Can be defined as an product term that is 1 in exactly one row of the truth table
- Maxterm:
  - An n-variable maxterm is a normal sum term with n literals.
  - There are 2^n such sum terms.
  - Examples of 4-variable maxterms:

    $W' + X' + Y + Z'$ $W + X' + Y' + Z$ $W' + X' + Y + Z$
  - Can be defined as a sum term that is 0 in exactly one row in the truth table.

Minterms/Maxterms for a 3-variable function

Row	X	Y	Z	F	*Minterm*	*Maxterm*
0	0	0	0	F(0,0,0)	$X' \cdot Y' \cdot Z'$	$X + Y + Z$
1	0	0	1	F(0,0,1)	$X' \cdot Y' \cdot Z$	$X + Y + Z'$
2	0	1	0	F(0,1,0)	$X' \cdot Y \cdot Z'$	$X + Y' + Z$
3	0	1	1	F(0,1,1)	$X' \cdot Y \cdot Z$	$X + Y' + Z'$
4	1	0	0	F(1,0,0)	$X \cdot Y' \cdot Z'$	$X' + Y + Z$
5	1	0	1	F(1,0,1)	$X \cdot Y' \cdot Z$	$X' + Y + Z'$
6	1	1	0	F(1,1,0)	$X \cdot Y \cdot Z'$	$X' + Y' + Z$
7	1	1	1	F(1,1,1)	$X \cdot Y \cdot Z$	$X' + Y' + Z'$

Canonical Sum Representation

- Minterm i :
  - Row i of the truth table that have output 1
- Canonical sum:
  - Sum of all minterms for a given function (truth table)
- The Σ notation:
  - Ex: $\Sigma_{X,Y,Z}(0, 3, 4, 6, 7)$
    $= X'\cdot Y'\cdot Z' + X'\cdot Y\cdot Z + X\cdot Y'\cdot Z' + X\cdot Y\cdot Z' + X\cdot Y\cdot Z$
  - This representation is usually realized using 2-level AND-OR logic circuits with inverters at AND gates inputs as needed

Canonical Sum Example

- The function represented by the truth table:

Row	X	Y	Z	F
0	0	0	0	1
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

has the canonical sum representation:

$F = \Sigma_{X,Y,Z}(0, 3, 4, 6, 7)$ ← **Minterm list using Σ notation**

$= X'\cdot Y'\cdot Z' + X'\cdot Y\cdot Z + X\cdot Y'\cdot Z' + X\cdot Y\cdot Z' + X\cdot Y\cdot Z$

Algebraic canonical sum of minterms

Canonical Product Representation

- Maxterm i:
  - Row i of the truth table that have output 0
- Canonical product:
  - Product of the maxterms for a given function (truth table)
- The Π notation:
  - Ex: $\Pi_{X,Y,Z}(1,2,5)$
    $= (X + Y + Z') . (X + Y' + Z) . (X' + Y + Z')$
  - This representation is usually realized using 2-level OR-AND logic circuits with inverters at OR gates inputs as needed.

Canonical Product Example

- The function represented by the truth table:

Row	X	Y	Z	F
0	0	0	0	1
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

has the canonical product representation:

$F = \Pi_{X,Y,Z}(1,2,5)$ ← **Maxterm list using Π notation**

$= (X + Y + Z') \cdot (X + Y' + Z) \cdot (X' + Y + Z')$

Algebraic canonical product of maxterms

Converting between Minterm/Maxterm Lists

- Find the complement of the set …
- Examples:

$$\Sigma_{X,Y,Z}(0,1,2,3) = \Pi_{X,Y,Z}(4,5,6,7)$$

$$\Sigma_{X,Y}(1) = \Pi_{X,Y}(0,2,3)$$

$$\Sigma_{W,X,Y,Z}(0,1,2,3,5,7,11,13) = \Pi_{W,X,Y,Z}(4,6,8,9,12,14,15)$$